Reduction Formulae 102

(vi) Evaluate $\int_0^{\pi/2} \cos^n x \, dx$

$$\int_0^{\pi/2} \cos^n x \, dx = \int_0^{\pi/2} \cos^n\left(\frac{\pi}{2} - x\right) dx$$

$$= \int_0^{\pi/2} \sin^n x \, .$$

Depending upon 'n' is even or odd.

SUNDAY

$$\boxed{\begin{aligned} \int_0^{\pi/2} \cos^n x \, dx &= \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} \cdots \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2} \quad \text{for even} \\ &= \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} \cdots \frac{4}{5} \cdot \frac{2}{3} \quad \text{for odd.} \end{aligned}}$$

12 MONDAY

Week 42 ■ 286-080

Ex Evaluate $\int_0^1 \frac{x^6 \, dx}{\sqrt{1-x^2}}$

Soln Substituting $x = \sin\theta \Rightarrow dx = \cos\theta \, d\theta$

If $x = 0 \Rightarrow \sin\theta = 0 \Rightarrow \theta = 0$

If $x = 1 \Rightarrow \sin\theta = 1 \Rightarrow \theta = \pi/2$

$$\therefore I = \int_0^{\pi/2} \frac{\sin^6\theta \cos\theta \, d\theta}{\sqrt{1-\sin^2\theta}} = \int_0^{\pi/2} \frac{\sin^6\theta \cos\theta \, d\theta}{\cos\theta}$$

$$= \int_0^{\pi/2} \sin^6\theta \, d\theta$$

Since $n = 6$

therefore $\displaystyle\int_0^{\pi/2} \sin^6\theta \, d\theta = \frac{6-1}{6} \cdot \frac{6-3}{6-2} \cdot \frac{6-5}{6-4} \cdot \frac{\pi}{2}$

$$= \frac{5}{6} \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2}$$

$$= \frac{5\pi}{32} \quad \underline{\text{Ans}}$$

(vii) $\displaystyle\int_0^{\pi/2} \sin^m x \cos^n x \, dx \; ; \; m, n > 0 \;\&\; m, n \in I$

Such Integrals are denoted as $I_{m,n}$

Case I

If 'n' is odd integer then it doesn't matter whether 'm' is even or odd, formula to be used is same

i.e., $\boxed{I_{m,n} = \frac{n-1}{m+n} \cdot \frac{n-3}{m+n-2} \cdots \frac{2}{m+3} \cdot \frac{1}{m+1}}$

Case II If 'n' is even and 'm' is even then

$$\boxed{I_{m,n} = \frac{(m-1)(m-3)\ldots 5 \cdot 3 \cdot 1 \cdot (n-1)(n-3)\ldots 5 \cdot 3 \cdot 1}{(m+n)(m+n-2)\ldots 4 \cdot 2} \cdot \frac{\pi}{2}}$$

If 'n' is even and 'm' is odd then

$$\boxed{I_{m,n} = \frac{(m-1)(m-3)\ldots 4 \cdot 2 \, (n-1)(n-3)\ldots 3 \cdot 1}{(m+n)(m+n-2)\ldots 5 \cdot 3 \cdot 1}}$$

Ex Evaluate $\int_0^{\pi/2} \sin^6 x \cos^4 x \, dx$

Soln In the given integral both m and n are even and $m = 6$, $n = 4$.

$$\therefore I_{m,n} = \frac{(6-1)(6-3)(6-5)(4-1)(4-3)}{(6+4)(6+4-2)(6+4-4)(6+4-6)(6+4-8)} \cdot \frac{\pi}{2}$$

$$\Rightarrow I_{6,4} = \frac{5\cdot 3\cdot 1\cdot 3\cdot 1}{2\cdot 10\cdot 8\cdot 6\cdot 4\cdot 2} \cdot \frac{\pi}{2}$$

$$= \frac{3\pi}{512} \quad \text{Ans}$$

Ex Evaluate $\int_0^{\pi/2} \sin^3\theta \cos^4\theta \, d\theta$

Soln Here 'n' is odd and $n = 3$, $m = 4$

$$\therefore I_{3,4} = \frac{3-1}{3+4} \cdot \frac{1}{4+1} = \frac{2}{7} \cdot \frac{1}{5}$$

$$= \frac{2}{35}$$

Ans

Ex Prove that $\int_0^{\pi/2} \sin^4 x \cos^5 x \, dx = \frac{8}{315}$

Soln $LHS = \int_0^{\pi/2} \sin^4 x \cos^5 x \, dx$

Here m is even and n is odd & $m=4, n=5$

$\therefore I_{4,3} = \frac{5-1}{4+5} \cdot \frac{5-3}{5+4-2} \cdot \frac{1}{4+1}$

$= \frac{4}{9} \cdot \frac{2}{7} \cdot \frac{1}{5}$

$= \frac{8}{615} = RHS$

$\Rightarrow LHS = RHS$

Proved